

ℓ^2 BOUNDED VARIATION AND ABSOLUTELY CONTINUOUS SPECTRUM OF JACOBI MATRICES

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ABSTRACT. We disprove a conjecture of Breuer–Last–Simon [1] concerning the absolutely continuous spectrum of Jacobi matrices with coefficients that obey an ℓ^2 bounded variation condition with step q . We prove existence of a.c. spectrum on a smaller set than that specified by the conjecture and prove that our result is optimal.

1. INTRODUCTION

In this paper we study semi-infinite Jacobi matrices

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & a_3 & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

where $a_n > 0$, $b_n \in \mathbb{R}$. We assume that

$$\sup_n a_n^{-1} + \sup_n a_n + \sup_n |b_n| < \infty, \quad (1.1)$$

in which case J is a bounded self-adjoint operator on $\ell^2(\mathbb{N})$. A canonical spectral measure μ corresponds to the cyclic vector δ_1 through

$$\int x^n d\mu(x) = \langle \delta_1, J^n \delta_1 \rangle, \quad n = 0, 1, 2, \dots$$

and if the Lebesgue decomposition of μ is

$$d\mu = f(x)dx + d\mu_s,$$

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we will be interested in the essential support of the a.c. spectrum,

$$\Sigma_{\text{ac}}(J) = \{x \in \mathbb{R} \mid f(x) > 0\}.$$

This set should properly be viewed as an equivalence class of sets modulo sets of Lebesgue measure zero. The absolutely continuous spectrum of J is then equal to the essential closure of $\Sigma_{\text{ac}}(J)$, defined as the set of $x \in \mathbb{R}$ such that $|\Sigma_{\text{ac}}(J) \cap (x - \epsilon, x + \epsilon)| > 0$ for all $\epsilon > 0$; see [9] for an expository discussion.

For ℓ^2 perturbations of coefficients of the free Jacobi matrix, [4, 12] proved $\Sigma_{\text{ac}}(J) = [-2, 2]$. Their sum rule approach initiated a search for higher-order Szegő theorems for Jacobi [15, 23, 13, 14] and CMV matrices [27, 25, 10, 19, 21, 7, 8, 2], in which ℓ^2 bounded variation conditions are combined with slow decay conditions (ℓ^p for some $p > 2$) to prove presence of a.c. spectrum on the spectrum of the free case. In this paper, we consider the implications of an ℓ^2 bounded variation condition without any decay conditions.

This paper focuses on Jacobi matrices such that for some $q \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} |a_{n+q} - a_n|^2 + \sum_{n=1}^{\infty} |b_{n+q} - b_n|^2 < \infty. \quad (1.2)$$

The implications of condition (1.2) on $\Sigma_{\text{ac}}(J)$ have been the subject of a series of papers and conjectures of various levels of generality, relating the a.c. spectrum of J to the a.c. spectra of its right limits. A two-sided Jacobi matrix $J^{(r)}$ with coefficients $a_n^{(r)} > 0$, $b_n^{(r)} \in \mathbb{R}$, $n \in \mathbb{Z}$ is called a right limit of J if there is a sequence $n_j \in \mathbb{Z}$, $n_j \rightarrow +\infty$, such that for all $n \in \mathbb{Z}$,

$$\lim_{j \rightarrow \infty} a_{n+n_j} = a_n^{(r)}, \quad \lim_{j \rightarrow \infty} b_{n+n_j} = b_n^{(r)}.$$

When (1.1) holds, a compactness argument shows that J has at least one right limit; the same argument shows that for every sequence $n_j \rightarrow +\infty$ there exists a subsequence which gives rise to a right limit. We will denote the set of right limits of J by \mathcal{R} . We are interested in the following conjecture from [1].

Conjecture 1.1 ([1, Conjecture 9.5]). *Let $q \in \mathbb{N}$ and let (1.2) hold. Then*

$$\Sigma_{\text{ac}}(J) = \bigcap_{\mathcal{R}} \sigma(J^{(r)}). \quad (1.3)$$

A narrower version of this conjecture, for $a_n \equiv 1$ and $b_n \rightarrow 0$, was previously made by Last [16] and proven by Denisov [6]. Further work of Kaluzhny–Shamis [11] proved (1.3) in the case where the sequences $\{a_n\}$, $\{b_n\}$ are asymptotically periodic (so there is, up to shifts, only one

right limit). These results have been carried over to orthogonal polynomials on the unit circle and extended beyond asymptotic periodicity by one of the authors [20]. Additional motivation for the conjecture is provided by work of Denisov [5] for Schrödinger operators, which can be seen as a continuum analog of Corollary 1.3 below.

However, we will construct examples which show that Conjecture 1.1 is false for $q > 1$. We will also prove a result which establishes a.c. spectrum on a smaller set and our examples will show that this result is optimal. This will also imply Conjecture 1.1 for $q = 1$.

The condition (1.2) implies

$$\lim_{n \rightarrow \infty} |a_{n+q} - a_n| = \lim_{n \rightarrow \infty} |b_{n+q} - b_n| = 0, \quad (1.4)$$

which implies that all right limits of J are q -periodic, since

$$a_{n+q}^{(r)} - a_n^{(r)} = \lim_{j \rightarrow \infty} (a_{n+q+n_j} - a_{n+n_j}) = 0$$

and analogously $b_{n+q}^{(r)} - b_n^{(r)} = 0$. The discriminant of a two-sided q -periodic Jacobi matrix $J^{(r)}$ is defined as

$$\Delta^{(r)}(z) = \text{tr} \left(A(a_q^{(r)}, b_q^{(r)}; z) A(a_{q-1}^{(r)}, b_{q-1}^{(r)}; z) \dots A(a_1^{(r)}, b_1^{(r)}; z) \right) \quad (1.5)$$

where tr denotes trace and $A(a, b; z)$ is the transfer matrix

$$A(a, b; z) = \begin{pmatrix} \frac{z-b}{a} & -\frac{1}{a} \\ a & 0 \end{pmatrix}. \quad (1.6)$$

It is well known [24, Chapter 5] that for such a Jacobi matrix,

$$\sigma(J^{(r)}) = \sigma_{\text{ac}}(J^{(r)}) = \{x \in \mathbb{R} \mid \Delta^{(r)}(x) \in [-2, 2]\} \quad (1.7)$$

and that this set is, in a natural way, a union of q closed intervals (“bands”) in \mathbb{R} whose interiors are disjoint. One can naturally define the q -interior of the spectrum as the union of interiors of the q bands, which can be expressed as

$$q\text{-int}(\sigma(J^{(r)})) = \{x \in \mathbb{R} \mid \Delta^{(r)}(x) \in (-2, 2)\}.$$

Note that this notion depends on q and cannot be expressed solely in terms of $J^{(r)}$, in the sense that a q -periodic Jacobi matrix can also be viewed as $2q$ -periodic, $3q$ -periodic, etc, and $q\text{-int}(\sigma(J^{(r)}))$, $(2q)\text{-int}(\sigma(J^{(r)}))$, $(3q)\text{-int}(\sigma(J^{(r)})) \dots$ are all distinct sets.

We can now state the theorem.

Theorem 1.2. *Let (1.1) and (1.2) hold for some $q \in \mathbb{N}$. Then*

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) \subset \Sigma_{\text{ac}}(J) \subset \bigcap_{\mathcal{R}} \sigma(J^{(r)}). \quad (1.8)$$

Moreover, for any closed interval

$$I \subset \bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})), \quad (1.9)$$

we have

$$\int_I \log f(x) dx > -\infty. \quad (1.10)$$

The second inclusion in (1.8) is, in fact, a general result of Last–Simon [17] for a.c. spectra of right limits, repeated here only for completeness. The essence of this theorem is in the first inclusion.

Although $q\text{-int}(\sigma(J^{(r)}))$ differs from $\sigma(J^{(r)})$ by only a finite set of points, those points can vary from right limit to right limit, so we would like to emphasize that the intersections in (1.8) can differ significantly. We will soon see examples of this.

For $q = 1$, the two inclusions of the previous theorem combine to give an equality.

Corollary 1.3. *If (1.1) holds and (1.2) holds for $q = 1$, then*

$$\Sigma_{\text{ac}}(J) = [\limsup_{n \rightarrow \infty} (b_n - 2a_n), \liminf_{n \rightarrow \infty} (b_n + 2a_n)]. \quad (1.11)$$

Moreover, (1.10) holds for each closed interval $I \subset \text{int}(\Sigma_{\text{ac}}(J))$.

Remark 1.1. This corollary sometimes yields intervals with purely singular spectrum. A result of Last–Simon [18, Theorem 3.1] for essential spectra of right limits implies

$$\sigma_{\text{ess}}(J) = [\liminf_{n \rightarrow \infty} (b_n - 2a_n), \limsup_{n \rightarrow \infty} (b_n + 2a_n)]$$

which can be strictly greater than the set (1.11), so the complement supports a purely singular part of the measure.

Another case in which the sets in (1.8) are equal is the case of convergence to an isospectral torus. This notion is the natural generalization of decaying perturbations of the free case; see, e.g., Last–Simon [18] and Damanik–Killip–Simon [3].

For our purposes, it suffices to define it as follows. Let $\mathcal{S} = \sigma(\tilde{J})$ for some q -periodic two-sided Jacobi matrix \tilde{J} . The isospectral torus of \mathcal{S} , denoted $\mathcal{T}_{\mathcal{S}}$, is the set of all q -periodic two-sided Jacobi matrices whose spectrum is equal to \mathcal{S} . It is known that this set is a k -dimensional torus for some $k \leq q - 1$, and that all elements of the isospectral torus have the same discriminant, which we will denote by $\Delta_{\mathcal{S}}(x)$.

We will say that J converges to the isospectral torus $\mathcal{T}_{\mathcal{S}}$ if all of its right limits lie on $\mathcal{T}_{\mathcal{S}}$. Of course, this generalizes asymptotic periodicity.

By [18], convergence of J to the isospectral torus $\mathcal{T}_{\mathcal{S}}$ implies $\sigma_{\text{ess}}(J) = \mathcal{S}$. With our ℓ^2 bounded variation condition (1.2), we can also say that $\sigma_{\text{ac}}(J) = \mathcal{S}$. More precisely, we have:

Corollary 1.4. *Let (1.1) and (1.2) hold for some $q \in \mathbb{N}$. If $\{a_n, b_n\}_{n=1}^{\infty}$ converges to an isospectral torus $\mathcal{T}_{\mathcal{S}}$, then*

$$\Sigma_{\text{ac}}(J) = \mathcal{S}. \quad (1.12)$$

Moreover, (1.10) holds for any closed interval $I \subset \mathcal{S}$ such that $|\Delta_{\mathcal{S}}(x)| < 2$ for all $x \in I$.

By Corollary 1.3, Conjecture 1.1 is true for $q = 1$. For an arbitrary $q > 1$, we will now discuss examples in which the two intersections in (1.8) are distinct and $\Sigma_{\text{ac}}(J)$ is equal to one or the other. This will show that, in general, no better statement can be made than (1.8).

Our examples will be taken from the class of discrete Schrödinger operators ($a_n \equiv 1$). Moreover, let us choose a parameter $\lambda \in (0, 2)$ and assume that the set of right limits is the set of constant Jacobi matrices with $a_n \equiv 1$ and $b_n \equiv \beta$ for $\beta \in [-\lambda, \lambda]$,

$$\mathcal{R} = \{J(1, \beta) \mid \beta \in [-\lambda, \lambda]\}. \quad (1.13)$$

The q -discriminant of the free Jacobi matrix $J(1, 0)$ is

$$\Delta(z) = \text{tr} \left(\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}^q \right)$$

and it is well known [24] that

$$\sigma(J(1, 0)) = [-2, 2]$$

and

$$q\text{-int}(\sigma(J(1, 0))) = (-2, 2) \setminus \{z_1, \dots, z_{q-1}\}$$

where z_1, \dots, z_{q-1} are the distinct solutions of $\Delta'(z) = 0$,

$$z_j = 2 \cos \left(\frac{(q-j)\pi}{q} \right).$$

Since $J(1, \beta)$ is just $J(1, 0) + \beta$, it follows that

$$\bigcap_{\mathcal{R}} \sigma(J^{(r)}) = [-2 + \lambda, 2 - \lambda]$$

and

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) = (-2 + \lambda, 2 - \lambda) \setminus \bigcup_{j=1}^{q-1} [z_j - \lambda, z_j + \lambda].$$

As promised, these intersections are distinct for $\lambda \in (0, 2)$. Moreover, we may have

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) = \emptyset$$

even when $\bigcap_{\mathcal{R}} \sigma(J^{(r)})$ is a fairly large interval. (In fact, for any positive λ , $\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)}))$ is empty if q is large enough.) Now we will see that each can be the essential support of the a.c. spectrum for a suitable Jacobi matrix (where empty essential support means there is no a.c. spectrum). The first of these two theorems disproves Conjecture 1.1.

Theorem 1.5. *Let $q \in \mathbb{N}$, $q > 1$. There exists a half-line Jacobi matrix J with the properties (1.1), (1.2) and with $a_n \equiv 1$ such that its set of right limits is the set \mathcal{R} given by (1.13) and*

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) = \Sigma_{\text{ac}}(J) \neq \bigcap_{\mathcal{R}} \sigma(J^{(r)}).$$

Theorem 1.6. *Let $q \in \mathbb{N}$, $q > 1$. There exists a half-line Jacobi matrix J with the properties (1.1), (1.2) and with $a_n \equiv 1$ such that its set of right limits is the set \mathcal{R} given by (1.13) and*

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) \neq \Sigma_{\text{ac}}(J) = \bigcap_{\mathcal{R}} \sigma(J^{(r)}).$$

The rest of this paper is organized as follows. In Sections 2 and 3, we prove Theorem 1.2, using the method of Denisov [6] and Kaluzhny–Shamis [11] together with some adaptations first made in [20] in the OPUC setting. In Section 4, we apply it to Corollaries 1.3 and 1.4. In Section 5 we prove Theorem 1.5 using a method from [16]. In Section 6 we prove Theorem 1.6.

2. ESTIMATES AND DIAGONALIZATION OF q -STEP TRANSFER MATRICES

We denote the q -step transfer matrix between positions mq and $(m+1)q$ and its trace and entries by

$$\Phi_m(z) = A(a_{(m+1)q}, b_{(m+1)q}; z) A(a_{(m+1)q-1}, b_{(m+1)q-1}; z) \dots A(a_{mq+1}, b_{mq+1}; z)$$

$$\Delta_m(z) = \text{tr } \Phi_m(z)$$

$$\Phi_m(z) = \begin{pmatrix} A_m(z) & B_m(z) \\ C_m(z) & D_m(z) \end{pmatrix}$$

In this section, we prepare for the proof of Theorem 1.2 by establishing certain properties of $\Phi_m(z)$ which will be needed later. They are mostly uniform estimates, necessary because without asymptotic periodicity of Jacobi parameters, we do not have convergence of $\Phi_m(z)$ in m . They

are analogs of estimates made in [20] for orthogonal polynomials on the unit circle.

The following are standard facts about q -step transfer matrices [24, Chapter 5].

- Theorem 2.1.** (i) $\det \Phi_m(z) = 1$;
(ii) $z \in \mathbb{R}$ implies $A_m(z), B_m(z), C_m(z), D_m(z), \Delta_m(z), \Delta'_m(z) \in \mathbb{R}$;
(iii) $\Delta_m(z) \in [-2, 2]$ implies $z \in \mathbb{R}$;
(iv) $\Delta_m(z) \in (-2, 2)$ implies $\Delta'_m(z) \neq 0$;
(v) $\Delta_m(z) \in (-2, 2)$ implies $C_m(z) \neq 0$.

Although the notation $\Phi_m(z)$ is convenient, we find it useful to think about $\Phi_m(z)$ as a fixed (m -independent) function of

$$a_{mp+1}, a_{mp+2}, \dots, a_{(m+1)p} \in (0, \infty), \quad b_{mp+1}, b_{mp+2}, \dots, b_{(m+1)p} \in \mathbb{R}, \quad z \in \mathbb{C}.$$

In that point of view, note that $\Phi_m(z)$ is an analytic function of its parameters, and the same is true of $A_m(z), B_m(z), C_m(z), D_m(z)$ and $\Delta_m(z)$. For any such function $f_m(z)$, if (1.1) holds, then for any compact $K \subset \mathbb{C}$, analyticity and compactness imply that there is a constant $C < \infty$ such that for all $m \geq 0$ and $z \in K$,

$$|f_m(z)| \leq C, \tag{2.1}$$

$$|f_{m+1}(z) - f_m(z)| \leq C \sum_{k=1}^q (|a_{(m+1)q+k} - a_{mq+k}| + |b_{(m+1)q+k} - b_{mq+k}|). \tag{2.2}$$

For $z \in \mathbb{C}$, let us define

$$L(z) = \limsup_{m \rightarrow \infty} |\Delta_m(z)|.$$

Lemma 2.2. Assume (1.1) and (1.4). Then $L(z)$ is finite for all $z \in \mathbb{C}$,

$$L(z) = \max_{\mathcal{R}} |\Delta^{(r)}(z)|,$$

$L(z)$ is Lipschitz continuous on any compact subset of \mathbb{C} , and

$$\bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})) = \{x \in \mathbb{R} \mid L(x) < 2\}, \tag{2.3}$$

which is an open set.

This lemma follows easily from compactness arguments and the observation that it suffices to consider right limits stemming from a sequence of n_j which are divisible by q . For more details, compare with Lemma 3.2 in [20].

The basic structure of the proof of Theorem 1.2 is to pick a closed interval I with the property (1.9) and prove (1.10). To prove (1.10),

we will need some uniform estimates which hold on such an interval. By (2.3) and continuity of L ,

$$\max_{x \in I} L(x) < 2.$$

Lemma 2.3 (analogous to [20, Lemma 3.3]). *Assume (1.1) and (1.4) and let $I \subset \mathbb{R}$ be a closed interval such that (1.9) holds. Then there exist $m_0 \in \mathbb{N}_0$, $s, t \in \{-1, +1\}$, $\epsilon \in (0, 1)$ and $C > 0$ such that for all $m \geq m_0$ and $z \in \Omega$,*

$$|\Delta_m(z)| \leq 2 - C \quad (2.4)$$

$$-s \operatorname{Re} \Delta'_m(z) \geq C \quad (2.5)$$

$$C \leq t \operatorname{Re} C_m(z) \leq |C_m(z)| \leq C^{-1} \quad (2.6)$$

where

$$\Omega = \{x + iy \mid x \in I, y \in [0, \epsilon]\}. \quad (2.7)$$

Our next goal is to diagonalize the $\Phi_m(z)$ for $m \geq m_0$ and $z \in \Omega$ in a way which obeys certain uniform estimates in z and m . To do this, we choose an eigenvalue of $\Phi_m(z)$ in a consistent way. With s as in (2.5), define

$$\lambda_m(z) = \frac{\Delta_m(z) \pm is\sqrt{4 - \Delta_m(z)^2}}{2},$$

where we take the branch of $\sqrt{\cdot}$ on $\mathbb{C} \setminus (-\infty, 0]$ such that $\sqrt{1} = 1$.

Lemma 2.4. *$\lambda_m(z)$ and $\lambda_m^{-1}(z)$ are the eigenvalues of $\Phi_m(z)$, and they obey the following estimates for some $C > 0$, uniformly in $m \geq m_0$, $z \in \Omega$:*

$$C \leq -s \operatorname{Im} \lambda_m(z) \leq |\lambda_m(z)| \leq 1 - C \operatorname{Im} z \quad (2.8)$$

$$s \operatorname{Im} \lambda_m^{-1}(z) \geq C. \quad (2.9)$$

Proof. $\lambda_m(z)$ and $\lambda_m^{-1}(z)$ are eigenvalues of $\Phi_m(z)$ since $\det \Phi_m(z) = 1$ and $\operatorname{tr} \Phi_m(z) = \Delta_m(z)$. Note that

$$\frac{\partial}{\partial y} \Delta_m(x + iy) = i \Delta'_m(x + iy)$$

so, taking imaginary parts and multiplying by s ,

$$s \frac{\partial}{\partial y} \operatorname{Im} \Delta_m(x + iy) = s \operatorname{Re} \Delta'_m(x + iy) \leq -C$$

for some $C > 0$ independent of m and $x + iy \in \Omega$, by (2.5). Integrating in y and using $\operatorname{Im} \Delta_m(x) = 0$,

$$s \operatorname{Im} \Delta_m(x + iy) = \int_0^y s \frac{\partial}{\partial y} \operatorname{Im} \Delta_m(x + it) dt \leq -Cy. \quad (2.10)$$

By Lemma 4.1 of [20],

$$|\lambda_m(z)| \leq 1 + s \operatorname{Im} \Delta_m(x + iy). \quad (2.11)$$

Combining (2.10) and (2.11), we obtain the upper bound on $|\lambda_m(z)|$ in (2.8). The bounds on $s \operatorname{Im} \lambda_m^{\pm 1}(z)$ follow from Lemma 4.1(iii) of [20]. \square

We now diagonalize $\Phi_m(z)$ as

$$\Phi_m(z) = U_m(z) \Lambda_m(z) U_m(z)^{-1} \quad (2.12)$$

where

$$\Lambda_m(z) = \begin{pmatrix} \lambda_m(z) & 0 \\ 0 & \lambda_m^{-1}(z) \end{pmatrix}$$

and

$$U_m(z) = \begin{pmatrix} \lambda_m(z) - D_m(z) & \lambda_m^{-1}(z) - D_m(z) \\ C_m(z) & C_m(z) \end{pmatrix}. \quad (2.13)$$

We chose columns of $U_m(z)$ to be eigenvectors of $\Phi_m(z)$, ensuring (2.12). Note that $\det U_m = (\lambda_m - \lambda_m^{-1})C_m \neq 0$ by (2.6) and Lemma 2.4. We also compute

$$U_m^{-1} = \frac{1}{(\lambda_m - \lambda_m^{-1})C_m} \begin{pmatrix} C_m & D_m - \lambda_m^{-1} \\ -C_m & \lambda_m - D_m \end{pmatrix} \quad (2.14)$$

and define

$$W_m = U_m^{-1} U_{m+1} - I.$$

By (2.2) and the preceding discussion, it is clear that

$$\|U_{m+1} - U_m\| \leq C \sum_{k=1}^q (|a_{(m+1)q+k} - a_{mq+k}| + |b_{(m+1)q+k} - b_{mq+k}|).$$

Together with $\|U_m^{-1}\| \leq C$, this implies that

$$\|W_m\| \leq C \sum_{k=1}^q (|a_{(m+1)q+k} - a_{mq+k}| + |b_{(m+1)q+k} - b_{mq+k}|)$$

for some value of $C < \infty$, uniformly in $m \geq m_0$ and $z \in \Omega$, and so by (1.2),

$$\sum_{m=0}^{\infty} \|W_m\|^2 < \infty. \quad (2.15)$$

3. PROOF OF THEOREM 1.2

In this section we conclude the proof of Theorem 1.2, adapting the method of Denisov [6] and Kaluzhny–Shamis [11].

Our first step is to follow an idea of [11] of introducing approximants of J which are eventually periodic and relating the a.c. parts of their spectral measures to certain Weyl solutions. For [11], the coefficients in their approximants were eventually equal to the periodic background; since we are working without asymptotic periodicity, we instead extend by periodicity from some point on.

Therefore, we define the Jacobi matrix J^N , $N = 0, 1, \dots$, so that its first $(N + 1)q$ Jacobi coefficients agree with those of J , and extending the sequence of coefficients by q -periodicity after that; i.e., the Jacobi coefficients of J^N are

$$a_{mq+r}^N = a_{\min(m,N)q+r}, \quad m \in \mathbb{N}_0, \quad r = 1, \dots, q \quad (3.1)$$

$$b_{mq+r}^N = b_{\min(m,N)q+r}, \quad m \in \mathbb{N}_0, \quad r = 1, \dots, q \quad (3.2)$$

We will also use the superscript N to denote other quantities corresponding to J^N ; for instance, the q -step transfer matrices corresponding to J^N are, by (3.1) and (3.2),

$$\Phi_m^N(z) = \Phi_{\min(N,m)}(z).$$

For $z \in \Omega$ and $N \geq m_0$, we wish to single out a solution $u^N(z)$ of the transfer matrix recursion,

$$u_{n+1}^N(z) = \Phi_n^N(z)u_n^N(z).$$

This is a first order recurrence relation, so since all Φ_n are invertible, we can specify the solution by setting its value at $n = N$,

$$u_N^N(z) = \begin{pmatrix} \lambda_N(z) - D_N(z) \\ C_N(z) \end{pmatrix}. \quad (3.3)$$

Let μ^N , the canonical spectral measure of J^N , have the Lebesgue decomposition

$$d\mu^N = f^N dx + d\mu_s^N.$$

We can now describe f^N in terms of u^N . This is a rewriting of equation (3.5) of [11]. We deviate cosmetically from [11] in using a solution of the transfer matrix recursion rather than a solution of the Jacobi recursion. We prefer this point of view because it avoids a need to extend the Jacobi recursion to the endpoint $n = 0$ and because it clarifies the analogy with the case of orthogonal polynomials on the unit circle covered in [20].

Lemma 3.1. *Let $N \geq m_0$. For every $x \in I$, $(u_0^N)_2(x) \neq 0$. For Lebesgue-a.e. $x \in I$,*

$$f^N(x) = -\frac{C_N(x) \operatorname{Im} \lambda_N(x)}{\pi |(u_0^N)_2(x)|^2}. \quad (3.4)$$

Remark 3.1. By Theorem 2.1(ii), we already know that the right hand side of (3.4) is real-valued. In fact, using the above formula and comparing $f^N(x) \geq 0$ with (2.6) and (2.8) gives $s = t$, but that observation will not be needed in what follows.

Proof. For $x \in \mathbb{R}$, the matrices $\Phi_n(x)$ have real entries, so $\overline{u^N(x)}$ is also a solution of the same recursion. By the constancy of their Wronskian (see, e.g., [24, Prop. 3.2.3]),

$$(u_0^N)_1(x) \overline{(u_0^N)_2(x)} - (u_0^N)_2(x) \overline{(u_0^N)_1(x)} = (u_N^N)_1(x) \overline{(u_N^N)_2(x)} - (u_N^N)_2(x) \overline{(u_N^N)_1(x)},$$

which, using (3.3) and Theorem 2.1(ii), simplifies to

$$\operatorname{Im}((u_0^N)_1(x) \overline{(u_0^N)_2(x)}) = \operatorname{Im}((u_N^N)_1(x) \overline{(u_N^N)_2(x)}) = C_N(x) \operatorname{Im} \lambda_N(x). \quad (3.5)$$

In particular, by (2.6) and (2.8), this implies that $(u_0^N)_1(x) \overline{(u_0^N)_2(x)} \neq 0$ for $x \in I$.

For $z \in \Omega \setminus I$, from $\Phi_n^N u_N^N = \lambda_N u_N^N$ for $n \geq N$ and $|\lambda_N| < 1$ it follows that u_n^N is a Weyl solution (see, e.g., [24, Section 3.2]). Thus, $u_0^N(z)$ is a multiple of $\begin{pmatrix} m^N(z) \\ -1 \end{pmatrix}$, where m^N is the Weyl m -function for J^N .

Thus,

$$m(z) = -\frac{(u_0^N)_1(z)}{(u_0^N)_2(z)}.$$

For almost every $x \in \mathbb{R}$, the nontangential limit of $\operatorname{Im} m^N(x)$ is equal to $\pi f^N(x)$, so

$$f^N(x) = -\frac{1}{\pi} \lim_{\epsilon \downarrow 0} \operatorname{Im} \frac{(u_0^N)_1(x + i\epsilon)}{(u_0^N)_2(x + i\epsilon)}.$$

The limit exists for all $x \in I$ because u_N^N , and so u_n^N for every n , is continuous in $z \in \Omega$. Using (3.5), this simplifies to (3.4). \square

Coefficient stripping is the operation of removing the leading Jacobi coefficients from the Jacobi matrix, i.e. replacing the sequence of coefficients $\{a_n, b_n\}_{n=1}^\infty$ by $\{a_n, b_n\}_{n=2}^\infty$. This operation does not affect the validity of conclusions of Theorem 1.2, so we perform coefficient stripping finitely many times and prove the result for the Jacobi matrix obtained in this way, from which the result for the original Jacobi matrix will follow.

Thus, in the following we may assume that all the above estimates, derived for $m \geq m_0$, now hold for all $m \geq 0$, and that, instead of (2.15),

$$\sum_{n=0}^{\infty} \|W_n\|^2 < \delta \quad (3.6)$$

for a suitably chosen $\delta > 0$.

The recursion relation for u_n^N , solved backwards, gives

$$u_0^N = \tilde{\Phi}_0^{-1} \cdots \tilde{\Phi}_{N-1}^{-1} u_N^N.$$

Using the diagonalization of $\tilde{\Phi}_n$ and computing $U_N^{-1} u_N^N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, this becomes

$$U_0^{-1} u_0^N = \Lambda_0^{-1} (I + W_0) \cdots \Lambda_{N-1}^{-1} (I + W_{N-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (3.7)$$

Let us label the entries of $W_n(z)$,

$$W_n(z) = \begin{pmatrix} E_n(z) & F_n(z) \\ G_n(z) & H_n(z) \end{pmatrix}.$$

From (2.13) and (2.14) we compute

$$\begin{aligned} 1 + E_n &= \frac{C_n(\lambda_{n+1} - D_{n+1}) + C_{n+1}(D_n - \lambda_n^{-1})}{(\lambda_n - \lambda_n^{-1})C_n}, \\ 1 + H_n &= \frac{-C_n(\lambda_{n+1}^{-1} - D_{n+1}) + C_{n+1}(\lambda_n - D_n)}{(\lambda_n - \lambda_n^{-1})C_n}. \end{aligned}$$

We will need the inequalities

$$\left| \log \prod_{n=k}^l |1 + E_n| \right| \leq C + Cv\sqrt{l-k} \quad (3.8)$$

$$\left| \log \prod_{n=k}^l |1 + H_n| \right| \leq C + Cv\sqrt{l-k} \quad (3.9)$$

with $v = \text{Im } z$ and with a constant C independent of $z \in \Omega$. This is proved almost as in the proof of Theorem 2.2 of [6]; a modification is needed where [6] uses convergence of coefficients, so Lemma 2.5 of [6] must be replaced by Lemma 6.2 of [20].

We now have all the estimates needed to apply a theorem of Denisov [6], made precisely to estimate such expressions.

Theorem 3.2 ([6, Theorem 2.1]). *Assume that (3.7) holds, that*

$$C > |\lambda_n^{-1}| > \kappa > 1 \quad (3.10)$$

and that (3.6) for a sufficiently small δ . Assume also there is a constant $v \in [0, 1)$ such that (3.8), (3.9) hold. Then there is a value of $C_1 \in (0, \infty)$, which depends only on C , such that

$$U_0^{-1}u_0^N = \prod_{j=0}^{N-1} (\lambda_j^{-1}(1 + E_j)) \begin{pmatrix} \phi_N \\ \nu_N \end{pmatrix} \quad (3.11)$$

where

$$|\phi_N|, |\nu_N| \leq C_1 \exp \left(\frac{C_1}{\kappa - 1} \exp \left(\frac{C_1 v^2}{\kappa - 1} \right) \right) \quad (3.12)$$

Moreover, for any fixed $\epsilon > 0$ and $\kappa > 1 + \epsilon$, we have

$$|\phi_N| > C_1^{-1} > 0, \quad |\nu_N| < C_1 \sum_{j=0}^{\infty} \|W_j\|^2 \quad (3.13)$$

uniformly in N .

By (2.8), this theorem is applicable to our case, with $\kappa = 1 + C \operatorname{Im} z$ and $v = \operatorname{Im} z$. and we conclude that (3.11) holds. By (3.12) and since $v^2/(\kappa - 1) = \operatorname{Im} z/C$ is uniformly bounded for $z \in \Omega$, ϕ_N, ν_N obey

$$|\phi_N|, |\nu_N| \leq \exp \left(\frac{C}{\operatorname{Im} z} \right) \quad (3.14)$$

for some $C < \infty$ and all N and $z \in \Omega$. Moreover, if δ in (3.6) has been chosen small enough, then by (3.13),

$$|\phi_N| > C, \quad |\nu_N| < \frac{C}{2}, \quad \text{for } z \in \Omega \text{ with } \operatorname{Im} z > \frac{\epsilon}{2}. \quad (3.15)$$

Multiplying (3.11) by $U_0(z)$ and using (2.13), we see

$$(u_0^N)_2(z) = \prod_{n=1}^N (\lambda_n^{-1}(z)(1 + E_n(z))) C_0(z)(\phi_N(z) + \nu_N(z)) \quad (3.16)$$

which we rewrite as

$$-\log |(u_0^N)_2(z)| = -\log \prod_{n=1}^N |\lambda_n^{-1}(z)(1 + E_n(z))| - \log |C_0(z)| + g_N(z) \quad (3.17)$$

where

$$g_N(z) = -\log |\phi_N(z) + \nu_N(z)|.$$

The above estimates imply the following lemma (the proof is analogous to the proof of Lemma 6.3 of [20]).

Lemma 3.3. *The function $g_N(z)$ is continuous on Ω and harmonic on $\text{int } \Omega$. There is a value of $C \in (0, \infty)$, independent of $N \in \mathbb{N}_0$, such that*

(i) *for all $x \in I$ and $N \in \mathbb{N}_0$,*

$$|\log f^N(x) - 2g_N(x)| \leq C \quad (3.18)$$

(ii) *for all $N \in \mathbb{N}_0$,*

$$\int_I g_N^+(x) dx \leq C \quad (3.19)$$

(iii) *for all $z \in \Omega \setminus I$ and $N \in \mathbb{N}_0$,*

$$g_N(z) \geq -\frac{C}{\text{Im } z} \quad (3.20)$$

(iv) *for all $z \in \Omega$ with $\text{Im } z > \frac{1}{2}\epsilon$ and $N \in \mathbb{N}_0$,*

$$g_N(z) \leq C. \quad (3.21)$$

We will also need the following lemma.

Lemma 3.4 ([6], [11, Lemma 2]). *Assume that $f(z)$ is continuous on Ω , harmonic on $\text{int } \Omega$, and for some $C, \alpha > 0$,*

$$\int_I g^+(x) dx < C,$$

$g(x+iy) > -Cy^{-\beta}$ for $x+iy \in \text{int } \Omega$, and $g(x+iy) < C$ for $x+iy \in \Omega$ with $y > \frac{C}{1+\beta}$. Then there is a constant B , depending only on C, β , so that

$$\int_I g^-(x) dx < B.$$

By Lemma 3.3, Lemma 3.4 is applicable to $g_N(z)$, and proves

$$\int_I g_N(x) dx > C$$

with a constant C independent of N . By (3.19) and (3.18), this implies

$$\int_I \log f^N(x) dx > C$$

with a constant C independent of N .

This integral is a relative entropy. Since J^N converge strongly to J , the measures μ^N converge weakly to μ , so by upper semicontinuity of entropy [24, Theorem 2.2.3],

$$\int_I \log f(x) dx \geq \limsup_{N \rightarrow \infty} \int_I \log f^N(x) dx \geq C > -\infty$$

which proves (1.10). Thus, $\log f(x) > -\infty$, and thus $f(x) > 0$, for a.e. $x \in I$.

Note that by (2.5), for any x in the set

$$S = \bigcap_{\mathcal{R}} q\text{-int}(\sigma(J^{(r)})),$$

all right limits have the same sign of $(\Delta^{(r)})'(x)$. Let J be a band in the spectrum of some right limit of J . Then $(\Delta^{(r)})'(x)$ have constant sign for all right limits and all $x \in J \cap S$, so $J \cap S$ is an interval or the empty set. Since this is true for any of the q bands, we see that S is the union of at most q open intervals.

Thus, S can be written as a countable union of closed intervals I . By the above, for each such I , $\{x \in I \mid f(x) = 0\}$ has zero Lebesgue measure, so we conclude that $\{x \in S \mid f(x) = 0\}$ has zero Lebesgue measure and the first inclusion of (1.8) follows. The second inclusion of (1.8) is a general result of Last–Simon [17], which completes the proof of Theorem 1.2.

4. PROOFS OF COROLLARIES 1.3 AND 1.4

Proof of Corollary 1.3. In this case all right limits are 1-periodic, with $a_n^{(r)} = \alpha^{(r)}$ and $b_n^{(r)} = \beta^{(r)}$ for some $\alpha^{(r)} > 0$ and $\beta^{(r)} \in \mathbb{R}$. For such a right limit, by (1.5) and (1.7),

$$\sigma(J^{(r)}) = [\beta^{(r)} - 2\alpha^{(r)}, \beta^{(r)} + 2\alpha^{(r)}]$$

and

$$1\text{-int}(\sigma(J^{(r)})) = (\beta^{(r)} - 2\alpha^{(r)}, \beta^{(r)} + 2\alpha^{(r)}).$$

Since every sequence of n_j has a subsequence which gives rise to a right limit, denoting $A_{\pm} = \pm \liminf_{n \rightarrow \infty} (2a_n \pm b_n)$, (1.8) becomes

$$(-A_-, A_+) \subset \Sigma_{\text{ac}}(J) \subset [-A_-, A_+].$$

The difference between the left and right hand sides is a finite set, which is negligible since $\Sigma_{\text{ac}}(J)$ is only defined up to a set of Lebesgue measure zero, so (1.11) follows. \square

Proof of Corollary 1.4. Since all right limits are q -periodic and have the same spectrum \mathcal{S} , they have the same discriminant $\Delta_{\mathcal{S}}(x)$ (see, e.g., [24, Chapter 5]), so (1.8) becomes

$$\{x \in \mathbb{R} \mid \Delta_{\mathcal{S}}(x) \in (-2, 2)\} \subset \Sigma_{\text{ac}}(J) \subset \{x \in \mathbb{R} \mid \Delta_{\mathcal{S}}(x) \in [-2, 2]\}.$$

Since $\Delta_{\mathcal{S}}$ is a nontrivial polynomial, the difference between the left and right hand sides is a finite set, and (1.12) follows as in the previous proof. \square

5. PROOF OF THEOREM 1.5

To prove this theorem, we will rely on a method from [16]. The sequence b_n will be constructed out of two parts,

$$b_n = \lambda_n + W_n,$$

where λ_n will be a piecewise constant sequence which will oscillate between $-\lambda$ and λ , and W_n will be a product of a piecewise constant decaying sequence and a periodic sequence.

To construct W_n , we will pick a sequence of integers

$$0 = L_1 < L_2 < \dots, \quad (5.1)$$

a q -periodic sequence $\{V_n\}_{n=1}^\infty$ with

$$V_1 = V_2 = \dots = V_{q-1} = 0, \quad V_q = 1$$

and a decaying sequence w_j with $w_1 \leq 1$,

$$w_1 > w_2 > \dots \rightarrow 0.$$

Then we choose

$$W_n = w_l V_n, \quad L_l < n \leq L_{l+1}.$$

Note that this makes $\{W_n\}$ q -periodic between L_l and L_{l+1} . It is immediate that

$$\lim_{n \rightarrow \infty} W_n = 0$$

and

$$\sum_{n=1}^{\infty} |W_{n+q} - W_n|^2 \leq q \sum_{l=1}^{\infty} |w_{l+1} - w_l|^2 < \infty.$$

To construct the sequence λ_n , we will refine the partition (5.1) by choosing a sequence $\{m_l\}_{l=1}^\infty$ such that

$$m_l \geq 2^l$$

and, for each $l \in \mathbb{N}$, a sequence of integers

$$L_l = n_{l,0} < n_{l,1} < \dots < n_{l,m_l} = L_{l+1}.$$

Then we will pick λ_n to be constant between $n_{l,k}$ and $n_{l,k+1}$,

$$\lambda_n = (-1)^l \left(1 - \frac{2k}{m_l}\right) \lambda, \quad n_{l,k} < n \leq n_{l,k+1}$$

It is then straightforward to check that

$$\limsup_{n \rightarrow \infty} \lambda_n = \lambda, \quad \liminf_{n \rightarrow \infty} \lambda_n = -\lambda,$$

and

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|^2 \leq 4\lambda^2 \sum_{l=1}^{\infty} \frac{1}{m_l} \leq 4\lambda^2 \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty.$$

It follows from the above that such a Jacobi matrix has the properties (1.1), (1.2) and the correct set of right limits, so Theorem 1.2 implies that

$$(-2 + \lambda, 2 - \lambda) \setminus \bigcup_{j=1}^{q-1} [z_j - \lambda, z_j + \lambda] \subset \Sigma_{\text{ac}}(J) \subset [-2 + \lambda, 2 - \lambda]. \quad (5.2)$$

Therefore, to prove Theorem 1.5, it suffices to show that we can choose the parameters $\{L_l\}$, $\{m_l\}$ and $\{n_{l,k}\}$ consistently with the above constraints, in such a way that there is no a.c. spectrum on $(z_j - \lambda, z_j + \lambda)$, for $j = 1, \dots, q - 1$.

This will be accomplished with the help of the following two propositions from [16], which, as pointed out there, follow from [17]. We denote by $T_{m,n}(x)$ the transfer matrix from m to n , i.e.

$$T_{m,n}(x) = A(a_n, b_n; x) A(a_{n-1}, b_{n-1}; x) \dots A(a_m, b_m; x).$$

Proposition 5.1. *For a.e. $x \in \Sigma_{\text{ac}}(J)$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N \log^2 N} \sum_{n=1}^N \|T_{1,n}(x)\|^2 < \infty. \quad (5.3)$$

Proposition 5.2. *Let J, \tilde{J} be discrete Schrödinger operators on $\ell^2(\mathbb{N})$. Suppose that for some $m, k \in \mathbb{N}$, $k > 4$, we have $b_n = \tilde{b}_n$ for $n \in \{m, m+1, \dots, k\}$, and that for some $E \in \mathbb{R}$ and $\delta > 0$, $\sigma(\tilde{J}) \cap (E - \delta, E + \delta) = \emptyset$. Then for $l \in \{4, 5, \dots, k\}$,*

$$\|T_{m,m+l}(E)\| \geq \frac{1}{2} \delta^2 (1 + \delta^2)^{\frac{l-3}{2}}.$$

Let us also note an obvious crude estimate which we will need later. Our Jacobi matrix has $a_n = 1$ and $|b_n| \leq \lambda + w_1 \leq 3$ for all n , so for $x \in \sigma(J) \subset [-5, 5]$,

$$\|A(1, b; x)\| \leq 2 + |x - b| \leq 10$$

which implies

$$\|T_{1,n}(x)\| \leq 10^n.$$

The idea of this construction is to have a sequence b_n which locally looks like a constant λ_n plus the periodic potential V_n with coupling constant w_l . As we slowly modulate λ_n , we will slowly move the gaps of $w_l V_n$, covering intervals of approximate length 2λ . By keeping a gap over a point x long enough (i.e. by making $n_{l,k+1} - n_{l,k}$ long enough),

we will be able to use Proposition 5.2 to show increase of norms of transfer matrices at x , which will contradict (5.3) and show absence of a.c. spectrum at x .

Therefore, the only property we need about the potential V_n is that for any positive value of the coupling constant, all gaps are open. For any $w > 0$, let us consider the q -periodic discrete Schrödinger operator J_w with diagonal terms wV_n .

Lemma 5.3. *For any $w > 0$, the discrete q -periodic Schrödinger operator J_w with potential wV_n has $q - 1$ open gaps.*

Proof. J_w is a q -periodic discrete Schrödinger operator, and its q -step transfer matrix is

$$\begin{aligned}\Phi(x) &= \begin{pmatrix} x - w & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}^{q-1} \\ &= \begin{pmatrix} p_q(x) - wp_{q-1}(x) & -p_{q-1}(x) + wp_{q-2}(x) \\ p_{q-1}(x) & -p_{q-2}(x) \end{pmatrix}\end{aligned}$$

where p_n are Chebyshev polynomials of the second kind, given by $p_n(2 \cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$. A closed gap at x would imply that $\Phi(x) = \pm I$, which would imply $p_{q-1}(x) = p_{q-2}(x) = 0$. These polynomials obey the recurrence relation $p_{n+1}(x) = xp_n(x) - p_{n-1}(x)$; using the recurrence relation backwards, this would imply $p_n(x) = 0$ for $n = q-3, q-4, \dots, 0$, which would contradict $p_0(x) = 1$. \square

For $j = 1, \dots, q-1$, denote by $z_{l,j}$ the center of the j -th gap of J_{w_l} . Let δ_l denote the minimum width of a gap of J_{w_l} and pick m_l so that $m_l \geq 4/\delta_l$.

Then λ_n oscillates from $-\lambda$ to λ with steps of size $\frac{2}{m_l} \leq \frac{\delta_l}{2}$, so for every

$$x \in \left[z_{l,j} - \lambda + \frac{3\delta_l}{4}, z_{l,j} + \lambda - \frac{3\delta_l}{4} \right], \quad (5.4)$$

there is a value of $k \in \{0, \dots, m_l - 1\}$ such that

$$\left| x - z_{l,j} - (-1)^l \left(1 - \frac{2k}{m_l} \right) \lambda \right| \leq \frac{\delta_l}{4},$$

i.e.

$$\left(x - \frac{\delta_l}{4}, x + \frac{\delta_l}{4} \right) \cap \sigma \left(J_{w_l} + (-1)^l \left(1 - \frac{2k}{m_l} \right) \lambda \right) = \emptyset$$

Since J coincides with $J_{w_l} + (-1)^l \left(1 - \frac{2k}{m_l}\right) \lambda$ at positions $n_{l,k}+1, \dots, n_{l,k+1}$, we can apply Proposition 5.2 to conclude

$$\begin{aligned} \sum_{n=1}^{n_{l,k+1}} \|T_{1,n}(x)\|^2 &\geq \sum_{n=n_{l,k}+5}^{n_{l,k+1}} \|T_{1,n_{l,k}}\|^{-2} \|T_{n_{l,k}+1,n}(x)\|^2 \\ &\geq 10^{-2n_{l,k}} \sum_{n=n_{l,k}+5}^{n_{l,k+1}} \|T_{n_{l,k}+1,n}(x)\|^2 \\ &\geq 10^{-2n_{l,k}} \sum_{n=n_{l,k}+5}^{n_{l,k+1}} \frac{1}{4} \left(\frac{\delta_l}{4}\right)^4 \left(1 + \left(\frac{\delta_l}{4}\right)^2\right)^{n-n_{l,k}-4} \end{aligned}$$

The right hand side grows exponentially as a function of $n_{l,k+1}$, so we can pick $n_{l,k+1}$ sufficiently large so that the right hand side is larger than $l n_{l,k+1} \log^2 n_{l,k+1}$. This will accomplish

$$\frac{1}{n_{l,k+1} \log^2 n_{l,k+1}} \sum_{n=1}^{n_{l,k+1}} \|T_{1,n}(x)\|^2 \geq l.$$

If we construct the $n_{l,k}$ inductively in this way, starting from $n_{l,0} = L_l$ and stopping at n_{l,m_l} , we will have for every x from (5.4),

$$\sup_{N \leq L_{l+1}} \frac{1}{N \log^2 N} \sum_{n=1}^N \|T_{1,n}(x)\|^2 \geq l. \quad (5.5)$$

As $l \rightarrow \infty$, $w_l \rightarrow 0$, so $\delta_l \rightarrow 0$ and $z_{l,j} \rightarrow z_j$ for $j = 1, \dots, q-1$. Thus, for any $x \in (z_j - \lambda, z_j + \lambda)$, (5.4) holds for sufficiently large l . Therefore, (5.5) also holds for large enough l , implying

$$\limsup_{N \rightarrow \infty} \frac{1}{N \log^2 N} \sum_{n=1}^N \|T_{1,n}(x)\|^2 = \infty.$$

By Proposition 5.1, this implies that there is no a.c. spectrum on $(z_j - \lambda, z_j + \lambda)$ for $j = 1, \dots, q-1$. Combined with (5.2), this implies

$$\Sigma_{\text{ac}}(J) = (-2 + \lambda, 2 - \lambda) \setminus \bigcup_{j=1}^{q-1} [z_j - \lambda, z_j + \lambda],$$

which completes the proof.

6. PROOF OF THEOREM 1.6

To construct a Jacobi matrix with the desired properties, it suffices to take $a_n \equiv 1$ and pick a sequence b_n such that

$$\begin{aligned}\limsup_{n \rightarrow \infty} b_n &= \lambda \\ \liminf_{n \rightarrow \infty} b_n &= -\lambda\end{aligned}$$

and

$$\sum_{n=1}^{\infty} |b_{n+1} - b_n|^2 < \infty.$$

For instance, we may choose

$$b_n = \lambda \cos(n^\gamma) \tag{6.1}$$

for $\gamma \in (0, \frac{1}{2})$. This clearly obeys (1.1) and (1.2) for the given q . But by Corollary 1.3,

$$\Sigma_{\text{ac}}(J) = [-2 + \lambda, 2 - \lambda], \tag{6.2}$$

which completes the proof.

We should remark here that the Jacobi matrix given by (6.1) is also in a class of slowly oscillating Jacobi matrices studied by Stolz [26], who proved (6.2) by different methods.

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